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June 2009
6669 Further Pure Mathematics FP3 (new)
Mark Scheme

Question Number	Scheme	Marks
Q1	$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5 \Rightarrow \frac{14}{e^x + e^{-x}} - \frac{(e^x - e^{-x})}{e^x + e^{-x}} = 5$	M1
	$\therefore 14 - (e^x - e^{-x}) = 5(e^x + e^{-x}) \Rightarrow 6e^x - 14 + 4e^{-x} = 0$	A1
	$\therefore 3e^{2x} - 7e^x + 2 = 0 \Rightarrow (3e^x - 1)(e^x - 2) = 0$	M1
	$\therefore e^x = \frac{1}{3}$ or 2	A1
	$x = \ln(\frac{1}{3})$ or $\ln 2$	B1ft **[5]**
Alternative (i)	Write 7 – sinh$x$ = 5cosh$x$, then use exponential substitution $7 - \frac{1}{2}(e^x - e^{-x}) = \frac{5}{2}(e^x + e^{-x})$ Then proceed as method above.	M1
Alternative (ii)	$(7\text{sech}x - 5)^2 = \tanh^2 x = 1 - \text{sech}^2 x$	M1
	$50\text{sech}^2 x - 70\text{sech}x + 24 = 0$	A1
	$2(5\text{sech}x - 3)(5\text{sech}x - 4) = 0$	M1
	sech$x$= $\frac{3}{5}$ or sech$x$= $\frac{4}{5}$	A1
	$x = \ln(\frac{1}{3})$ or $\ln 2$	B1ft
Q2 (a)	$\mathbf{b} \times \mathbf{c} = 0\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$	M1 A1 A1 (3)
(b)	$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0 + 5 = 5$	M1 A1 ft (2)
(c)	Area of triangle OBC = $\frac{1}{2}\lvert 5\mathbf{j} + 5\mathbf{k}\rvert = \frac{5}{2}\sqrt{2}$	M1 A1 (2)
(d)	Volume of tetrahedron = $\frac{1}{6} \times 5 = \frac{5}{6}$	B1 ft (1) **[8]**

Question Number	Scheme	Marks
Q3 (a)	$\begin{vmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \therefore (6-\lambda)(7-\lambda)(2-\lambda)+3(7-\lambda)=0$	M1
	$(7-\lambda)=0$ verifies $\lambda = 7$ is an eigenvalue (can be seen anywhere)	M1
	$\therefore (7-\lambda)\{12-8\lambda+\lambda^2+3\}=0 \quad \therefore (7-\lambda)\{\lambda^2-8\lambda+15\}=0$	A1
	$\therefore (7-\lambda)(\lambda-5)(\lambda-3)=0$ and 3 and 5 are the other two eigenvalues	M1 A1 (5)
(b)	Set $\begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	M1
	Solve $-x+y-z=0$ and $3x-y-5z=0$ to obtain $x = 3z$ or $y = 4z$ and a second equation which can contain 3 variables	M1 A1
	Obtain eigenvector as 3**i** + 4**j** + **k** (or multiple)	A1 (4) **[9]**

Question Number	Scheme	Marks
Q4 (a)	$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \times \frac{1}{\sqrt{1+(\sqrt{x})^2}}$	B1, M1
	$\therefore \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \frac{1}{2\sqrt{x(1+x)}}\right)$	A1 (3)
(b)	$\therefore \int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}dx = \left[2\text{ar}\sinh\sqrt{x}\right]_{\frac{1}{4}}^{4}$	M1
	$= \left[2\text{ar}\sinh 2 - 2\text{ar}\sinh(\tfrac{1}{2})\right]$	M1
	$= \left[2\ln(2+\sqrt{5})\right] - \left[2\ln(\tfrac{1}{2}+\sqrt{\tfrac{5}{4}})\right]$	M1
	$2\ln\frac{(2+\sqrt{5})}{(\frac{1}{2}+\sqrt{(\frac{5}{4})})} = 2\ln\frac{2(2+\sqrt{5})}{(1+\sqrt{5})} = 2\ln\frac{2(\sqrt{5}+2)(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = 2\ln\frac{(3+\sqrt{5})}{2}$	M1
	$= \ln\frac{(3+\sqrt{5})(3+\sqrt{5})}{4} = \ln\frac{14+6\sqrt{5}}{4} = \ln\left(\frac{7}{2}+\frac{3\sqrt{5}}{2}\right)$	A1 A1 (6) **[9]**
Alternative (i) for part (a)	Use $\sinh y = \sqrt{x}$ and state $\cosh y\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	B1
	$\therefore \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1+\sinh^2 y}} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{\left(1+\left(\sqrt{x}\right)^2\right)}}$	M1
	$\therefore \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \frac{1}{2\sqrt{x(1+x)}}\right)$	A1 (3)
Alternative (i) for part (b)	Use $x = \tan^2\theta$, $\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$ to give $2\int\sec\theta d\theta = \left[2\ln(\sec\theta+\tan\theta)\right]$	M1
	$= \left[2\ln(\sec\theta+\tan\theta)\right]_{\tan\theta=\frac{1}{2}}^{\tan\theta=2}$ i.e. use of limits	M1
	then proceed as before from line 3 of scheme	
Alternative (ii) for part (b)	Use $\int\frac{1}{\sqrt{[(x+\frac{1}{2})^2-\frac{1}{4}]}}dx = \text{arcosh}\frac{x+\frac{1}{2}}{\frac{1}{2}}$	M1
	$= \left[\text{arcosh}\,9 - \text{arcosh}(\tfrac{3}{2})\right]$	M1
	$= \left[\ln(9+\sqrt{80})\right] - \left[\ln(\tfrac{3}{2}+\tfrac{1}{2}\sqrt{5})\right]$	M1
	$= \ln\frac{(9+\sqrt{80})}{(\frac{3}{2}+\frac{1}{2}\sqrt{5})} = \ln\frac{2(9+\sqrt{80})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})},$	M1
	$= \ln\frac{2(9+4\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \ln\left(\frac{7}{2}+\frac{3\sqrt{5}}{2}\right)$	A1 A1 (6) **[9]**

Question Number	Scheme	Marks
Q5 (a)	$-(25-x^2)^{\frac{1}{2}}$ (+c)	M1A1 (2)
(b)	$I_n = \int x^{n-1}.\frac{x}{\sqrt{(25-x^2)}}\mathrm{d}x = -x^{n-1}\sqrt{25-x^2} + \int (n-1)x^{n-2}\sqrt{(25-x^2}\mathrm{d}x$	M1 A1ft
	$I_n = \left[-x^{n-1}\sqrt{25-x^2}\right]_0^5 + \int_0^5 \frac{(n-1)x^{n-2}(25-x^2)}{\sqrt{(25-x^2)}}\mathrm{d}x$	M1
	$I_n = 0 + 25(n\text{-}1)\ I_{n-2}\ \text{-}\ (n\text{-}1)\ I_n$	M1
	$\therefore nI_n = 25(n-1)I_{n-2}$ and so $I_n = \frac{25(n-1)}{n}I_{n-2}$ ✱	A1 (5)
(c)	$I_0 = \int_0^5 \frac{1}{\sqrt{(25-x^2}}\mathrm{d}x = \left[\arcsin(\frac{x}{5})\right]_0^5 = \frac{\pi}{2}$	M1 A1
	$I_4 = \frac{25\times 3}{4}\times\frac{25\times 1}{2}I_0 = \frac{1875}{16}\pi$	M1 A1 (4)
		[11]
Alternative for (b)	Using substitution $x = 5\sin\theta$	
	$I_n = 5^n\int_0^{\frac{\pi}{2}}\sin^n\theta\mathrm{d}\theta = \left[-5^n\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} + 5^n(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\cos^2\theta\mathrm{d}\theta$	M1A1
	$= \left[-5^n\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} + 5^n(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta(1-\sin^2\theta)\mathrm{d}\theta$	M1
	$I_n = 0 + 25(n\text{-}1)\ I_{n-2}\ \text{-}\ (n\text{-}1)\ I_n$	M1
	$\therefore nI_n = 25(n-1)I_{n-2}$ and so $I_n = \frac{25(n-1)}{n}I_{n-2}$ ✱	A1
	(need to see that $I_{n-2} = 5^{n-2}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\mathrm{d}\theta$ for final A1)	(5)

Question Number	Scheme	Marks
Q6 (a)	$\dfrac{x^2}{a^2} - \dfrac{(mx+c)^2}{b^2} = 1$ and so $b^2x^2 - a^2(mx+c)^2 = a^2b^2$	M1
	$\therefore\ (b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2+b^2) = 0$ Or $(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2+b^2) = 0$ ✳	A1 (2)
(b)	$(2a^2mc)^2 = 4(a^2m^2 - b^2) \times a^2(c^2+b^2)$	M1
	$4a^4m^2c^2 = -4a^2(b^2c^2 + b^4 - a^2m^2c^2 - a^2m^2b^2)$	
	$c^2 = a^2m^2 - b^2$ or $a^2m^2 = b^2 + c^2$ ✳	A1 (2)
(c)	Substitute (1, 4) into $y = mx+c$ to give $4 = m + c$ and	
	Substitute $a = 5$ and $b = 4$ into $c^2 = a^2m^2 - b^2$ to give $c^2 = 25m^2 - 16$	B1
	Solve simultaneous equations to eliminate m or c : $(4-m)^2 = 25m^2 - 16$	M1
	To obtain $24m^2 + 8m - 32 = 0$	A1
	Solve to obtain $8(3m+4)(m-1) = 0 ......m = ..\text{or}...$	M1
	$m = 1$ or $-\dfrac{4}{3}$	A1
	Substitute to get $c = 3$ or $\dfrac{16}{3}$	M1
	Lines are $y = x+3$ and $3y + 4x = 16$	A1 (7) **[11]**

Question Number	Scheme	Marks
Q7 (a)	If the lines meet, $-1+3\lambda=-4+3\mu$ and $2+4\lambda=2\mu$	M1
	Solve to give $\lambda=0$ $(\mu=1$ but this need not be seen$)$.	M1 A1
	Also $1-\lambda=\alpha$ and so $\alpha=1$.	B1 (4)
(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 0 & 3 & 2 \end{vmatrix}$ = $-6\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ is perpendicular to both lines and hence to the plane	M1 A1
	The plane has equation **r.n**=**a.n ,** which is $-6x + 2y - 3z = -14$,	M1
	i.e. $-6x + 2y - 3z + 14 = 0$.	A1 o.a.e. (4)
OR (b)	**Alternative scheme**	
	Use (1, -1,2) and (α ,-4,0) in equation $ax+by+cz+d=0$	M1
	And third point so three equations, and attempt to solve	M1
	Obtain $6x-2y+3z=$	A1
	$(6x-2y+3z)-14=0$	A1 o.a.e. (4)
(c)	$(\mathbf{a}_1 - \mathbf{a}_2) = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$	M1
	Use formula $\dfrac{(\mathbf{a}_1 - \mathbf{a}_2)\bullet\mathbf{n}}{\lvert\mathbf{n}\rvert} = \dfrac{(\mathbf{i}-3\mathbf{j}-2\mathbf{k})\bullet(-6\mathbf{i}+2\mathbf{j}-3\mathbf{k})}{\sqrt{(36+4+9)}} = \left(\dfrac{-6}{7}\right)$	M1
	Distance is $\dfrac{6}{7}$	A1 (3) **[11]**

Question Number	Scheme	Marks
Q8 (a)	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3\sin\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 5\cos\theta$	B1
	so S = $2\pi\int 5\sin\theta\sqrt{(-3\sin\theta)^2 + (5\cos\theta)^2}\,\mathrm{d}\theta$	M1
	$\therefore S = \quad 2\pi\int 5\sin\theta\sqrt{9 - 9\cos^2\theta + 25\cos^2\theta}\,\mathrm{d}\theta$	M1
	Let $c = \cos\theta$, $\frac{dc}{d\theta} = -\sin\theta$, limits 0 and $\frac{\pi}{2}$ become 1 and 0	M1
	So $S = k\pi\int_0^{\alpha}\sqrt{16c^2 + 9}\,\mathrm{d}c$, where k = 10, and α is 1	A1, A1 (6)
(b)	Let $c = \frac{3}{4}\sinh u$. Then $\frac{dc}{du} = \frac{3}{4}\cosh u$	M1
	So $S = k\pi\int_?^?\sqrt{9\sinh^2 u + 9}\;\frac{3}{4}\cosh u\,\mathrm{d}u$	A1
	$= k\pi\int_?^? \frac{9}{4}\cosh^2 u\,\mathrm{d}u = k\pi\int_?^? \frac{9}{8}(\cosh 2u + 1)\mathrm{d}u$	M1
	$= k\pi\left[\frac{9}{16}\sinh 2u + \frac{9}{8}u\right]_0^d$	A1
	$= \frac{45\pi}{4}\left[\frac{20}{9} + \ln 3\right]$ i.e. <u>117</u>	B1 (5)
		[11]

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Question Number	Scheme	Marks
1.	$\pm\dfrac{a}{e}=8, \quad \pm ae=2$	B1, B1
	$\dfrac{a}{e}\times ae=a^2=16$	
	$a=4$	B1
	$b^2=a^2(1-e^2)=a^2-a^2e^2$	
	$\Rightarrow b^2=16-4=12$	M1
	$\Rightarrow b=\sqrt{12}=2\sqrt{3}$	A1 (5)
		5

Question Number	Scheme	Marks
2.	$x^2+4x+13=(x+2)^2+9$	B1
	$\int \frac{1}{(x+2)^2+9}dx=\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)$	M1 A1
	$\left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1}=\frac{1}{3}\left(\arctan 1-\arctan 0\right)$	M1
	$=\frac{\pi}{12}$	A1 (5)
		5

Question Number	Scheme	Marks
3(a)	$rhs = 1 + 2\sinh^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$	M1
	$= \frac{2 + e^{2x} - 2 + e^{-2x}}{2}$	M1
	$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x = lhs$ *	A1 (3)
(b)	$1 + 2\sinh^2 x - 3\sinh x = 15$	
	$2\sinh^2 x - 3\sinh x - 14 = 0$	M1
	$(\sinh x + 2)(2\sinh x - 7) = 0$	M1
	$\sinh x = -2, \frac{7}{2}$	A1
	$x = \ln\left(-2 + \sqrt{((-2)^2 + 1)}\right) = \ln\left(-2 + \sqrt{5}\right)$	M1
	$x = \ln\left(\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 + 1)}\right) = \ln\left(\frac{7 + \sqrt{53}}{2}\right)$	A1 (5)
		8

Question Number	Scheme	Marks
4(a)	$\int (a-x)^n \cos x\mathrm{d}x = (a-x)^n \sin x + \int n(a-x)^{n-1} \sin x\mathrm{d}x$	M1A1
	$\left[(a-x)^n \sin x\right]_0^a = 0$	A1
	$= -n(a-x)^{n-1}\cos x - \int n(n-1)(a-x)^{n-2}\cos x\mathrm{d}x$	dM1
	$\mathrm{I}_n = na^{n-1} - n(n-1)\mathrm{I}_{n-2}$ *	A1 (5)
(b)	$\mathrm{I}_2 = 2\left(\frac{\pi}{2}\right) - 2\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x$	M1 A1
	$= \pi - 2\left[\sin x\right]_0^{\frac{\pi}{2}} = \pi - 2$	A1 (3)
		8

Question Number	Scheme	Marks
5(a)	$\frac{dy}{dx} = 2\text{ar}\cosh(3x) \times \frac{3}{\sqrt{9x^2-1}}$	M1A1A1
	$\sqrt{9x^2-1}\frac{dy}{dx} = 6\text{ar}\cosh(3x)$	
	$(9x^2-1)\left(\frac{dy}{dx}\right)^2 = 36(\text{ar}\cosh(3x))^2$	dM1
	$(9x^2-1)\left(\frac{dy}{dx}\right)^2 = 36y$ *	A1 (5)
(b)	$\left\{18x\left(\frac{dy}{dx}\right)^2 + (9x^2-1)\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}\right\} = 36\frac{dy}{dx}$	M1 {A1} A1
	$(9x^2-1)\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} = 18$ *	A1 (4)
		9

Question Number	Scheme	Marks
6(a)	$\begin{pmatrix}1&0&3\\0&-2&1\\k&0&1\end{pmatrix}\begin{pmatrix}6\\1\\6\end{pmatrix}=\lambda\begin{pmatrix}6\\1\\6\end{pmatrix}$	
	$\begin{pmatrix}24\\4\\6k+6\end{pmatrix}=\begin{pmatrix}6\lambda\\\lambda\\6\lambda\end{pmatrix}$	
	Uses the first or second row to obtain $\lambda=4$	M1A1 (2)
(b)	Uses the third row and their $\lambda=4$ to obtain $6k+6=24\Rightarrow k=3$ *	M1 A1 (2)
(c)	$\begin{vmatrix}1-\lambda&0&3\\0&-2-\lambda&1\\3&0&1-\lambda\end{vmatrix}=0$	
	$\Rightarrow(1-\lambda)\big((-2-\lambda)(1-\lambda)-0\big)-0\big(0(1-\lambda)-3\big)+3\big(0-3(-2-\lambda)\big)=0$	M1 A1
	$\Rightarrow(1-\lambda)(-2-\lambda)(1-\lambda)+9(2+\lambda)=(2+\lambda)\left(9-(1-\lambda)^2\right)=0$	
	$\left(\lambda^3-12\lambda-16=0\right)$	
	$\Rightarrow(\lambda+2)\left(\lambda^2-2\lambda-8\right)=0$	
	$\Rightarrow(\lambda+2)(\lambda+2)(\lambda-4)=0$	M1
	$\lambda=-2,4$	A1 (4)
(d)	Parametric form of l_1 : $(t+2,-3t,4t-1)$	M1
	$\begin{pmatrix}1&0&3\\0&-2&1\\3&0&1\end{pmatrix}\begin{pmatrix}t+2\\-3t\\4t-1\end{pmatrix}=\begin{pmatrix}13t-1\\10t-1\\7t+5\end{pmatrix}$	M1 A1
	Cartesian equations of l_2 : $\dfrac{x+1}{13}=\dfrac{y+1}{10}=\dfrac{z-5}{7}$	ddM1A1(5)
		13

Question Number	Scheme	Marks
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 0 \\ 6 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$	M1 A2(1,0)
	$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 5$	
	$\mathbf{r} \bullet \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$	M1A1 (5)
(b)	Equation of l is $\mathbf{r} = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$	M1
	At intersection $\begin{pmatrix} 6+t \\ 13+4t \\ 5+2t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$	M1
	$\Rightarrow 6+t+4(13+4t)+2(5+2t)=5 \Rightarrow t=-3$	M1
	N is $(3,1,-1)$ *	A1 (4)
(c)	$\overrightarrow{PN} \bullet \overrightarrow{PR} = (-3\mathbf{i}-12\mathbf{j}-6\mathbf{k}) \bullet (-5\mathbf{i}-13\mathbf{j}-3\mathbf{k}) = 189$	M1 A1ft
	$\sqrt{9+144+36}\sqrt{25+169+9}\cos NPR = 189$	A1
	$NX = NP\sin NPR = \sqrt{189}\sin NPR = 3.61$	M1A1 (5)
		14

Question Number	Scheme	Marks
8(a)	$\frac{dx}{dt} = 4\sec t \tan t \quad \frac{dy}{dt} = 2\sec^2 t$	B1 (both)
	$\frac{dy}{dx} = \frac{2\sec^2 t}{4\sec t \tan t} \quad \left(= \frac{1}{2\sin t}\right)$	M1
	$y - 2\tan t = \frac{1}{2\sin t}(x - 4\sec t)$	M1 A1
	$2y\sin t - \frac{4\sin^2 t}{\cos t} = x - \frac{4}{\cos t}$	
	$2y\sin t = x - \frac{4 - 4\sin^2 t}{\cos t} = x - 4\cos t \quad *$	A1 (5)
(b)	Gradient of l_2 is $-2\sin t$	M1
	$y = -2x\sin t \quad (2)$	A1
	$2(-2x\sin t)\sin t = x - 4\cos t \Rightarrow x = \frac{4\cos t}{1 + 4\sin^2 t} \quad (1)$	M1 A1
	$y = \frac{-8\sin t\cos t}{1 + 4\sin^2 t}$	M1 A1
	$(x^2 + y^2)^2 = \left(\frac{16\cos^2 t}{(1 + 4\sin^2 t)^2} + \frac{64\sin^2 t\cos^2 t}{(1 + 4\sin^2 t)^2}\right)^2$	
	$= \frac{256\cos^4 t}{(1 + 4\sin^2 t)^4}(1 + 4\sin^2 t)^2 = \frac{256\cos^4 t}{(1 + 4\sin^2 t)^2}$	M1
	$16x^2 - 4y^2 = \frac{256\cos^2 t}{(1 + 4\sin^2 t)^2} - \frac{256\sin^2 t\cos^2 t}{(1 + 4\sin^2 t)^2} = \frac{256\cos^4 t}{(1 + 4\sin^2 t)^2}$	A1 (8) **13**

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1.	$\frac{\text{dy}}{\text{dx}} = 6x^2$ and so surface area $= 2\pi\int 2x^3\sqrt{(1+(6x^2)^2}\,\text{dx}$	B1
	$= 4\pi\left[\frac{2}{3\times 36\times 4}(1+36x^4)^{\frac{3}{2}}\right]$	M1 A1
	Use limits 2 and 0 to give $\frac{4\pi}{216}[13860.016-1] = 806$ (to 3 sf)	DM1 A1
		5
	Notes:	
B1	Both bits CAO but condone lack of 2π	
1M1	Integrating $\int\left(y\sqrt{1+\left(\text{their}\frac{dy}{dx}\right)^2}\right)dx$, getting $k(1+36x^4)^{\frac{3}{2}}$, condone lack of 2π. If they use a substitution it must be a complete method.	
1A1	CAO	
2DM1	Correct use of 2 and 0 as limits	
2A1	CAO	
2. **(a) (i)**	$\frac{\text{dy}}{\text{dx}} = \frac{x}{\sqrt{(1-x^2)}} + \arcsin x$	M1 A1 (2)
(ii)	At given value derivative $= \frac{1}{\sqrt{3}} + \frac{\pi}{6} = \frac{2\sqrt{3}+\pi}{6}$	B1 (1)
(b)	$\frac{dy}{dx} = \frac{6e^{2x}}{1+9e^{4x}}$	1M1 A1
	$= \frac{6}{e^{-2x}+9e^{2x}}$	2M1
	$= \frac{3}{\frac{5}{2}(e^{2x}+e^{-2x})+\frac{4}{2}(e^{2x}-e^{-2x})}$	3M1
	$\therefore \frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}$ *	A1 cso
		(5) **8**
	Notes:	
(a) M1	Differentiating getting an arcsinx term and a $\frac{1}{\sqrt{1\pm x^2}}$ term	
A1	CAO	
B1	CAO any correct form	

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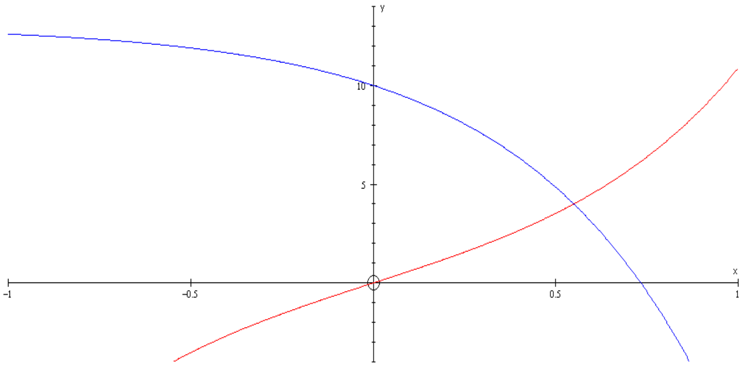
(b) 1M1	Of correct form $\frac{ae^{2x}}{1\pm be^{4x}}$	
1A1	CAO	
2M1	Getting from expression in e^{4x} to e^{2x} and e^{-2x} only	
3M1	**Using** sinh2x and cosh2x in terms of ($e^{2x}+e^{-2x}$) and ($e^{2x}-e^{-2x}$)	
2A1	CSO – answer given	
3. **(a)**	$x^2-10x+34=(x-5)^2+9$ so $\frac{1}{x^2-10x+34}=\frac{1}{(x-5)^2+9}=\frac{1}{u^2+9}$	B1
	(mark can be earned in either part (a) or (b))	
	$I=\int\frac{1}{u^2+9}du=\left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]$ | $I=\int\frac{1}{(x-5)^2+9}du=\left[\frac{1}{3}\arctan\left(\frac{x-5}{3}\right)\right]$	M1 A1
	Uses limits 3 and 0 to give $\frac{\pi}{12}$ | Uses limits 8 and 5 to give $\frac{\pi}{12}$	DM1 A1
		(5)
(b) Alt 1	$I=\ln\left(\left(\frac{x-5}{3}\right)+\sqrt{\left(\frac{x-5}{3}\right)^2+1}\right)$ or $I=\ln\left(\frac{x-5+\sqrt{(x-5)^2+9}}{3}\right)$	M1 A1
	or $I=\ln\left((x-5)+\sqrt{(x-5)^2+9}\right)$	
	Uses limits 5 and 8 to give $\ln(1+\sqrt{2})$.	DM1 A1
		(4)
		9
(b) Alt 2	Uses u = x-5 to get $I=\int\frac{1}{\sqrt{u^2+9}}du=\left[\text{ar}\sinh\left(\frac{u}{3}\right)\right]=\ln\left\{u+\sqrt{u^2+9}\right\}$	M1 A1
	Uses limits 3 and 0 and ln expression to give $\ln(1+\sqrt{2})$.	DM1 A1
		(4)
(b) Alt 3	Use substitution $x-5=3\tan\theta$, $\frac{dx}{d\theta}=3\sec^2\theta$ and so	M1 A1
	$I=\int\sec\theta d\theta=\ln(\sec\theta+\tan\theta)$	
	Uses limits 0 and $\frac{\pi}{4}$ to get $\ln(1+\sqrt{2})$.	DM1 A1
		(4)
	Notes:	
(a) B1	CAO allow recovery in (b)	
1M1	Integrating getting k arctan term	
1A1	CAO	
2DM1	Correctly using limits.	
2A1	CAO	

Question Number	Scheme	Marks
(b) 1M1	Integrating to get a ln or hyperbolic term	
1A1	CAO	
2DM1	Correctly using limits.	
2A1	CAO	
4.		
(a)	$I_n = \left[\frac{x^3}{3}(\ln x)^n\right] - \int \frac{x^3}{3} \times \frac{n(\ln x)^{n-1}}{x} dx$	M1 A1
	$= \left[\frac{x^3}{3}(\ln x)^n\right]_1^e - \int_1^e \frac{nx^2(\ln x)^{n-1}}{3} dx$	DM1
	$\therefore I_n = \frac{e^3}{3} - \frac{n}{3}I_{n-1}$ *	A1cso
		(4)
(b)	$I_0 = \int_1^e x^2 dx = \left[\frac{x^3}{3}\right]_1^e = \frac{e^3}{3} - \frac{1}{3}$ or $I_1 = \frac{e^3}{3} - \frac{1}{3}\left(\frac{e^3}{3} - \frac{1}{3}\right) = \frac{2e^3}{9} + \frac{1}{9}$	M1 A1
	$I_1 = \frac{e^3}{3} - \frac{1}{3}I_0$, $I_2 = \frac{e^3}{3} - \frac{2}{3}I_1$ and $I_3 = \frac{e^3}{3} - \frac{3}{3}I_2$ so $I_3 = \frac{4e^3}{27} + \frac{2}{27}$	M1 A1
		(4) **8**
	<u>Notes</u>:	
(a)1M1	Using integration by parts, integrating x^2, differentiating $(\ln x)^n$	
1A1	CAO	
2DM1	Correctly using limits 1 and e	
2A1	CSO answer given	
(b)1M1	Evaluating I_0 or I_1 by an attempt to integrate something	
1A1	CAO	
2M1	Finding I_3 (also probably I_1 and I_2) If 'n's left in M0	
2A1	I_3 CAO	

Question Number	Scheme	Marks
5. **(a)**	 Graph of y = 3sinh2x	B1
	Shape of $-e^{2x}$ graph	B1
	Asymptote: y = 13	B1
	Value 10 on y axis and value 0.7 or $\frac{1}{2}\ln\left(\frac{13}{3}\right)$ on x axis	B1
		(4)
(b)	Use definition $\frac{3}{2}(e^{2x}-e^{-2x})=13-3e^{2x} \rightarrow 9e^{4x}-26e^{2x}-3=0$ to form quadratic	M1 A1
	$\therefore e^{2x}=-\frac{1}{9}$ or 3	DM1 A1
	$\therefore x=\frac{1}{2}\ln(3)$	B1
		(5) **9**
	Notes:	
(a) 1B1	y = 3sinh2x first and third quadrant.	
2B1	Shape of y = $-e^{2x}$ correct intersects on positive axes.	
3B1	**Equation** of asymptote, y = 13, given. Penlise ‘extra’asymptotes here	
4B1	Intercepts correct both	
(b) 1M1	Getting a three terms quadratic in e^{2x}	
1A1	Correct three term quadratic	
2DM1	Solving for e^{2x}	
2A1	CAO for e^{2x} condone omission of negative value.	
B1	CAO one answer only	

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Question Number	Scheme	Marks
6. **(a)**	$\mathbf{n} = (2\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ o.a.e. (e.g. $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$)	M1 A1 (2)
(b)	Line l has direction $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ Angle between line l and normal is given by $(\cos\beta \text{ or } \sin\alpha) = \frac{4+2+2}{\sqrt{9}\sqrt{9}} = \frac{8}{9}$ $\alpha = 90 - \beta = 63$ degrees to nearest degree.	B1 M1 A1ft A1 awrt (4)
(c) Alt 1	Plane P has equation $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 1$ Perpendicular distance is $\frac{1-(-7)}{\sqrt{9}} = \frac{8}{3}$	M1 A1 M1 A1 (4) **10**
(c) Alt 2	Parallel plane through A has equation $\mathbf{r}.\frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{3} = \frac{-7}{3}$ Plane P has equation $\mathbf{r}.\frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{3} = \frac{1}{3}$ So O lies between the two and perpendicular distance is $\frac{1}{3} + \frac{7}{3} = \frac{8}{3}$	M1 A1 M1 A1 (4)
(c) Alt 3	Distance A to (3,1,2) = $\sqrt{2^2 + 2^2 + 1^2} = 3$ Perpendicular distance is '3' sin $\alpha = 3 \times \frac{8}{9} = \frac{8}{3}$	M1A1 M1A1 (4)
(c) Alt 4	Finding Cartesian equation of plane P: 2x – y – 2z – 1 = 0 $d = \frac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d\rvert}{\sqrt{n_1^2 + n_2^2 + n_3^2}} = \frac{\lvert 2(1) - 1(3) - 2(3) - 1\rvert}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{8}{3}$	M1 A1 M1A1 (4)
	Notes:	
(a) M1	Cross product of the correct vectors	
A1	CAO o.e.	
(b) B1	CAO	
M1	Angle between '$2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$' and $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, formula of correct form	
1A1ft	8/9ft	
2A1	CAO awrt	
(c) 1M1	Eqn of plane using $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ **or** dist of A from O **or** finding length of AP	
1A1	Correct equation (must have =) **or** A to (3,1,2) = 3	
2M1	Using correct method to find perpendicular distance	
2A1	CAO	

Question Number	Scheme	Marks
7. **(a)**	Det $\mathbf{M} = k(0-2)+1(1+3)+1(-2-0) = -2k+4-2 = 2-2k$	M1 A1 (2)
(b)	$\mathbf{M}^T = \begin{pmatrix} k & 1 & 3 \\ -1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ so cofactors = $\begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$	M1
	(-1 A mark for each term wrong) $\mathbf{M}^{-1} = \frac{1}{2-2k}\begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$	M1 A3 (5)
(c)	Let (x, y, z) be on l_1. Equation of l_2 can be written as $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$.	B1
	Use $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ with $k = 2$. i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2 & -1 & 1 \\ -4 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4+4\lambda \\ 1+\lambda \\ 7+3\lambda \end{pmatrix}$	M1
	$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\lambda+1 \\ 4\lambda-2 \\ 2\lambda \end{pmatrix}$	M1 A1
	and so $(\mathbf{r} - \mathbf{a})\times\mathbf{b} = \mathbf{0}$ where $\mathbf{a}$ = **i** - 2**j** and **b** = 3**i** +4 **j** + 2**k** or equivalent **or** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where **a** = **i** - 2**j** and **b** = 3**i** +4 **j** + 2**k** or equivalent	B1ft (5) **12**

Notes:

(a) M1	Finding determinant at least one component correct.
A1	CAO
(b) 1M1	Finding matrix of cofactors or its transpose
2M1	Finding inverse matrix, 1/(det) cofactors + transpose
1A1	At least seven terms correct (so at most 2 incorrect) condone missing det
2A1	At least eight terms correct (so at most 1 incorrect) condone missing det
3A1	All nine terms correct, condone missing det
(c) 1B1	Equation of l_2
1M1	Using inverse transformation matrix correctly
2M1	Finding general point in terms of λ.
A1	CAO for general point in terms of one parameter
2B1	ft for vector equation of their l_1

Question Number	Scheme	Marks
8. **(a)**	Uses $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cosh\theta}{a\sinh\theta}$ **or** $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \to y' = \frac{xb^2}{ya^2} = \frac{b\cosh\theta}{a\sinh\theta}$	M1 A1
	So $y - b\sinh\theta = \frac{b\cosh\theta}{a\sinh\theta}(x - a\cosh\theta)$	M1
	$\therefore ab(\cosh^2\theta - \sinh^2\theta) = xb\cosh\theta - ya\sinh\theta$ and as $(\cosh^2\theta - \sinh^2\theta) = 1$ $xb\cosh\theta - ya\sinh\theta = ab$ *	A1cso
		(4)
(b)	P is the point $(\frac{a}{\cosh\theta}, 0)$	M1 A1
		(2)
(c)	l_2 has equation $x = a$ and meets l_1 at Q $(a, \frac{b(\cosh\theta - 1)}{\sinh\theta})$	M1 A1
		(2)
(d) Alt 1	The mid point of PQ is given by $X = \frac{a(\cosh\theta + 1)}{2\cosh\theta}$, $Y = \frac{b(\cosh\theta - 1)}{2\sinh\theta}$	1M1 A1ft
	$4Y^2 + b^2 = b^2\left(\frac{\cosh^2\theta + 1 - 2\cosh\theta + \sinh^2\theta}{\sinh^2\theta}\right)$	2M1
	$= b^2\left(\frac{2\cosh^2\theta - 2\cosh\theta}{\sinh^2\theta}\right)$	3M1
	$X(4Y^2 + b^2) = ab^2\left(\frac{(\cosh\theta + 1)(\cosh\theta - 1)2\cosh\theta}{2\cosh\theta\sinh^2\theta}\right)$	4M1
	Simplify fraction by using $\cosh^2\theta - \sinh^2\theta = 1$ to give $x(4y^2 + b^2) = ab^2$ *	A1cso
		(6)
(d) Alt 2	First line of solution as before	1M1A1ft
	$4Y^2 + b^2 = b^2\left(\coth^2\theta + \text{cosech}^2\theta - 2\coth\theta\,\text{cosech}\theta + 1\right)$	2M1
	$= b^2\left(2\coth^2\theta - 2\coth\theta\,\text{cosech}\theta\right)$	3M1
	$X(4Y^2 + b^2) = ab^2\left(\coth\theta(\coth\theta - \text{cosech}\theta)\right)(1 + \text{sech}\theta))$	4M1
	Simplify expansion by using $\coth^2\theta - \text{cosech}^2\theta = 1$ to give $x(4y^2 + b^2) = ab^2$ *	A1cso
		(6)
		14

Question Number	Scheme	Marks
8.		
(a)1M1	Finding gradient in terms of θ	
1A1	CAO	
2M1	Finding equation of tangent	
2A1	CSO (answer given) look for $\pm(\cosh^2\theta - \sinh^2\theta)$	
(b)M1	Putting $y = 0$ into their tangent	
A1ft	P found, ft for their tangent o.e.	
(c) M1	Putting $x = a$ into their tangent.	
A1	CAO Q found o.e.	
(d)	For Alt 1 and 2	
1M1	Finding expressions, in terms of $\sinh\theta$ and $\cosh\theta$ but must be adding	
1A1	Ft on their P and Q,	
2M1	Finding $4y^2 + b^2$	
3M1	Simplified, factorised, maximum of 2 terms per bracket	
4M1	Finding $x(4y^2 + b^2)$, completely factorised, maximum of 2 terms per bracket	
2A1	CSO	
(d)	For Alts 3, 4 and 5	
1M1	Finding expressions, in terms of $\sinh\theta$ and $\cosh\theta$ but must be adding	
1A1	Ft on their P and Q	
2M1	Getting $\cosh\theta$ in terms of x	
3M1	y or y^2 in terms of $\cosh\theta$ **or** $\sinh\theta$ in terms of x and y	
4M1	Getting equation in terms of x and y only. No square roots.	
2A1	CSO	

Question Number	Scheme		Marks
8(d)			
Alt 3	$X = \dfrac{a(\cosh\theta+1)}{2\cosh\theta}, \quad Y = \dfrac{b(\cosh\theta-1)}{2\sinh\theta}$	As main scheme	1M1 A1ft
	$\cosh\theta = \dfrac{a}{2x-a}$	$\cosh\theta$ in terms of x	2M1
	$\sinh\theta = \dfrac{b(\cosh\theta-1)}{2y} = \dfrac{b(a-x)}{(2x-a)y}$	$\sinh\theta$ in terms of x and y	3M1
	$\left(\dfrac{a}{2x-a}\right)^2 - \left(\dfrac{b(a-x)}{(2x-a)y}\right)^2 = 1$	Using $\cosh^2\theta - \sinh^2\theta = 1$	4M1
	Simplifies to give required equation $\left[y^2 4x(a-x) = b^2(a-x)^2,\ x(4y^2+b^2) = ab^2\right]$		A1cso
			(6)
Alt 4	$X = \dfrac{a(\cosh\theta+1)}{2\cosh\theta}, \quad Y = \dfrac{b(\cosh\theta-1)}{2\sinh\theta}$	As main scheme	1M1 A1ft
	$\cosh\theta = \dfrac{a}{2x-a}$	$\cosh\theta$ in terms of x	2M1
	$y^2 = \dfrac{b^2(\cosh\theta - 1)^2}{4(\cosh^2\theta - 1)} = \dfrac{b^2(\cosh\theta - 1)}{4(\cosh\theta + 1)}$	y^2 in terms of $\cosh\theta$ only	3M1
	$y^2 = \dfrac{b^2\left(\dfrac{2a-2x}{2x-a}\right)^2}{4\left(\dfrac{2x}{2x-a}\right)}$ o.e	Forms equation in x and y only	4M1
	Simplifies to give required equation		A1 cso
			(6)
Alt 5	$X = \dfrac{a(\cosh\theta+1)}{2\cosh\theta}, \quad Y = \dfrac{b(\cosh\theta-1)}{2\sinh\theta}$	As main scheme	1M1 A1ft
	$\cosh\theta = \dfrac{a}{2x-a}$	$\cosh\theta$ in terms of x	2M1
	$y = \left(\dfrac{b(\cosh\theta - 1)}{2\sinh\theta}\right) = \left(\dfrac{b(\cosh\theta - 1)}{2\sqrt{\cosh^2\theta - 1}}\right)$	y in terms of $\cosh\theta$ only	3M1
	Eliminate $\sqrt{\ }$ and forms equation in x and y		4M1
	Simplifies to give required equation		A1cso

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6669 Further Pure Maths FP3
Mark Scheme

Question Number	Scheme	Marks
1. (a)	Uses formula to obtain $e = \frac{5}{4}$	M1A1
	Uses ae formula	M1 (3)
(b)	Uses other formula $\frac{a}{e}$	M1
	Obtains both Foci are $(\pm 5, 0)$ and Directrices are $x = \pm\frac{16}{5}$ (needs both method marks)	A1 cso (2) (5 marks)

Notes

a1M1: Uses $b^2 = a^2(e^2 - 1)$ to get $e > 1$

a1A1: cao

a2M1: Uses ae

b1M1: Uses $\frac{a}{e}$

b1A1: cso for both foci and both directrices. Must have both of the 2 previous M marks may be implicit.

Question Number	Scheme	Marks
2.	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh 3x$	B1
	so $s = \int \sqrt{1+\sinh^2 3x}\,\mathrm{d}x$	M1
	$\therefore s = \int \cosh 3x\,\mathrm{d}x$	A1
	$= \left[\frac{1}{3}\sinh 3x\right]_0^{\ln a}$	M1
	$= \frac{1}{3}\sinh 3\ln a = \frac{1}{6}[\mathrm{e}^{3\ln a} - \mathrm{e}^{-3\ln a}]$	DM1
	$= \frac{1}{6}(a^3 - \frac{1}{a^3})$ (so $k = 1/6$)	A1 (6 marks)

Notes

1B1: cao

1M1: Use of arc length formula, need both $\sqrt{\ }$ and $\left(\frac{dy}{dx}\right)^2$.

1A1: $\int \cosh 3x\mathrm{d}x$ cao

2M1: Attempt to integrate, getting a hyperbolic function o.e.

3M1: depends on previous M mark. Correct use of lna and 0 as limits. Must see some exponentials.

2A1: cao

Question Number	Scheme	Marks
3. (a)	$\overrightarrow{AC} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{BC} = -3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ $\overrightarrow{AC} \times \overrightarrow{BC} = 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}$	B1, B1 M1 A1 (4)
(b)	Area of triangle ABC = $\frac{1}{2}\lvert 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}\rvert = \frac{1}{2}\sqrt{1225} = 17.5$	M1 A1 (2)
(c)	Equation of plane is $10x - 15y + 30z = -20$ or $2x - 3y + 6z = -4$ So **r.** (2**i** – 3**j** + 6**k)** = -4 or correct multiple	M1 A1 (2) (8 marks)

Notes

a1B1: $\overrightarrow{AC} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ cao, any form

a2B1: $\overrightarrow{BC} = -3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ cao, any form

a1M1: Attempt to find cross product, modulus of one term correct.

a1A1: cao, any form.

b1M1: modulus of their answer to (a) – condone missing ½ here. To finding area of triangle by correct method.

b1A1: cao.

c1M1: [Using their answer to (a) to] find **equation** of plane. Look for **a.n** or **b.n** or **c.n** for p.

c1A1: cao

Question Number	Scheme	Marks
4(a)	$I_n = \left[x^n(-\tfrac{1}{2}\cos 2x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\tfrac{1}{2}nx^{n-1}\cos 2x\,dx$	M1 A1
	so $I_n = \left\langle \left[x^n(-\tfrac{1}{2}\cos 2x)\right]_0^{\frac{\pi}{4}} \right\rangle + \left[\tfrac{1}{4}nx^{n-1}\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tfrac{1}{4}n(n-1)x^{n-2}\sin 2x\,dx$	M1 A1
	i.e. $I_n = \frac{1}{4}n\left(\frac{\pi}{4}\right)^{n-1} - \frac{1}{4}n(n-1)I_{n-2}$ *	A1cso
		(5)
(b)	$I_0 = \int_0^{\frac{\pi}{4}} \sin 2x\,dx = \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} = \tfrac{1}{2}$	M1 A1
	$I_2 = \frac{1}{4}\times 2\times\left(\frac{\pi}{4}\right) - \frac{1}{4}\times 2\times I_0$, so $I_2 = \frac{\pi}{8} - \frac{1}{4}$	M1 A1 (4)
(c)	$I_4 = \left(\frac{\pi}{4}\right)^3 - \frac{1}{4}\times 4\times 3I_2 = \frac{\pi^3}{64} - 3\left(\frac{\pi}{8} - \frac{1}{4}\right) = \tfrac{1}{64}(\pi^3 - 24\pi + 48)$ *	M1 A1cso (2)

Notes

a1M1: Use of integration by parts, integrating $\sin 2x$, differentiating x^n.
a1A1: cao
a2M1: Second application of integration by parts, integrating $\cos 2x$, differentiating x^{n-1}.
a2A1: cao
a3A1: cso Including correct use of $\frac{\pi}{4}$ and 0 as limits.
b1M1: Integrating to find I_0 or setting up parts to find I_2.
b1A1: cao (Accept I_0 = ½ here for both marks)
b2M1: Finding I_2 in terms of π. If 'n''s left in M0
b2A1: cao
c1M1: Finding I_4 in terms of I_2 then in terms of π. If 'n''s left in M0
c1A1: cso

Question Number	Scheme	Marks
5. (a)	$\operatorname{ar}\sinh 2x, +x\dfrac{2}{\sqrt{1+4x^2}}$	M1A1, A1 (3)
(b)	$\therefore \int_0^{\sqrt{2}} \text{arsinh}2x\text{d}x = \left[x\operatorname{ar}\sinh 2x\right]_0^{\sqrt{2}} - \int_0^{\sqrt{2}} \dfrac{2x}{\sqrt{1+4x^2}}\text{d}x$ $= \left[x\operatorname{ar}\sinh 2x\right]_0^{\sqrt{2}} - \left[\dfrac{1}{2}(1+4x^2)^{\frac{1}{2}}\right]_0^{\sqrt{2}}$ $= \sqrt{2}\text{arsinh}2\sqrt{2} - [\frac{3}{2} - \frac{1}{2}]$ $= \sqrt{2}\ln(3+2\sqrt{2}) - 1$	1M1 1A1ft 2M1 2A1 3DM1 4M1 3A1 (7) (10 marks)

Notes

a1M1: Differentiating getting an arsinh term **and** a term of the form $\dfrac{px}{\sqrt{1\pm qx^2}}$

a1A1: cao $\operatorname{ar}\sinh 2x$

a2A1: cao + $\dfrac{2x}{\sqrt{1+4x^2}}$

b1M1: rearranging their answer to (a). **OR** setting up parts
b1A1: ft from their (a) **OR** setting up parts correctly
b2M1: Integrating getting an arsinh or arcosh term **and** a $\left(1\pm ax^2\right)^{\frac{1}{2}}$ term o.e..
b2A1: cao
b3DM1: depends on previous M, correct use of $\sqrt{2}$ and 0 as limits.
b4M1: converting to log form.
b3A1: cao depends on all previous M marks.

Question Number	Scheme	Marks
6(a)	$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ and so $\frac{dy}{dx}=-\frac{xb^2}{ya^2}=-\frac{b\cos\theta}{a\sin\theta}$	M1 A1
	$\therefore\ y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$	M1
	Uses $\cos^2\theta+\sin^2\theta=1$ to give $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ ✳	A1cso (4)
(b)	Gradient of circle is $-\frac{\cos\theta}{\sin\theta}$ and equation of tangent is	M1
	$y-a\sin\theta=-\frac{\cos\theta}{\sin\theta}(x-a\cos\theta)$ or sets $a=b$ in previous answer	
	So $y\sin\theta+x\cos\theta=a$	A1 (2)
(c)	Eliminate x or y to give $y\sin\theta(\frac{a}{b}-1)=0$ or $x\cos\theta(\frac{b}{a}-1)=b-a$	M1
	l_1 and l_2 meet at $(\frac{a}{\cos\theta}, 0)$	A1, B1 (3)
(d)	The locus of R is part of the line $y = 0$, such that $x \geq a$ and $x \leq -a$ Or clearly labelled sketch. Accept "real axis"	B1, B1 (2) (11 marks)

Notes

a1M1: Finding gradient in terms of θ. Must use calculus.
a1A1: cao
a2M1: Finding equation of tangent
a2A1: cso (answer given). Need to get $\cos^2\theta+\sin^2\theta$ on the same side.
b1M1: Finding gradient and equation of tangent, **or** setting $a = b$.
b1A1: cao need not be simplified.
c1M1: As scheme
c1A1: $x=\frac{a}{\cos\theta}$, need not be simplified.
c1B1: $y = 0$, need not be simplified.
d1B1: Identifying locus as $y = 0$ or real/'x' axis.
d2B1: Depends on previous B mark, identifies correct parts of $y = 0$. Condone use of strict inequalities.

Question Number	Scheme	Marks
7(a)	$\text{f}(x) = 5\cosh x - 4\sinh x = 5\times\frac{1}{2}(e^x + e^{-x}) - 4\times\frac{1}{2}(e^x - e^{-x})$	M1
	$= \frac{1}{2}(e^x + 9e^{-x})$ ∗	A1cso (2)
(b)	$\frac{1}{2}(e^x + 9e^{-x}) = 5 \Rightarrow e^{2x} - 10e^x + 9 = 0$	M1 A1
	So $e^x = 9$ or 1 and $x = \ln 9$ or 0	M1 A1 (4)
(c)	Integral may be written $\int \frac{2e^x}{e^{2x}+9}\,dx$	B1
	This is $\frac{2}{3}\arctan\left(\frac{e^x}{3}\right)$	M1 A1
	Uses limits to give $\left[\frac{2}{3}\arctan 1 - \frac{2}{3}\arctan(\frac{1}{\sqrt{3}})\right] = \left[\frac{2}{3}\times\frac{\pi}{4} - \frac{2}{3}\times\frac{\pi}{6}\right] = \frac{\pi}{18}$ ∗	DM1 A1cso (5)
		(11 marks)

Notes

a1M1: Replacing both cosh$x$ and sinh$x$ by terms in e^x and e^{-x} condone sign errors here.
a1A1: cso (answer given)
b1M1: Getting a three term quadratic in e^x
b1A1: cao
b2M1: solving to x =
b2A1: cao need ln9 (o.e) and 0 (not ln1)
c1B1: cao getting into suitable form, may substitute first.
c1M1: Integrating to give term in arctan
c1A1: cao
c2M1: Depends on previous M mark. Correct use of ln3 and ½ ln3 as limits.
c2A1: cso must see them subtracting two terms in π.

Question Number	Scheme	Marks
8. (a)	$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0 \therefore (2-\lambda)(2-\lambda)(4-\lambda)-(4-\lambda)=0$	M1
	$(4-\lambda)=0$ verifies $\lambda=4$ is an eigenvalue (can be seen anywhere)	M1
	$\therefore (4-\lambda)\left\{4-4\lambda+\lambda^2-1\right\}=0 \ \therefore (4-\lambda)\left\{\lambda^2-4\lambda+3\right\}=0$	A1
	$\therefore (4-\lambda)(\lambda-1)(\lambda-3)=0$ and 3 and 1 are the other two eigenvalues	M1 A1 (5)
(b)	Set $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	M1
	Solve $-2x+y=0$ and $x-2y=0$ and $-x=0$ to obtain $x = 0, y = 0, z = k$	M1
	Obtain eigenvector as **k** (or multiple)	A1 (3)
(c)	l_1 has equation which may be written $\begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$	B1
	So l_2 is given by **r** = $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$	M1
	i.e. **r** = $\begin{pmatrix} 8+\lambda \\ 7-\lambda \\ -11+7\lambda \end{pmatrix}$	M1 A1
	So $(\mathbf{r}-\mathbf{c})\times\mathbf{d}=\mathbf{0}$ where **c** = 8**i** + 7**j** - 11**k** and **d** = **i** - **j** + 7**k**	A1ft (5) (13 marks)

Notes

a1M1: Condone missing = 0. (They might expand the determinant using any row or column)
a2M1: Shows λ = 4 is an eigenvalue. Some working needed need to see = 0 at some stage.
a1A1: Three term quadratic factor cao, may be implicit (this A depends on 1st M only)
a2M1: Attempt at factorisation (usual rules), solving to λ = .
a2A1: cao. If they state λ = 1 and 3 please give the marks.
b1M1: Using A***x*** = 4***x*** o.e.
b2M1: Getting a pair of correct equations.
b1A1: cao
c1B1: Using **a** and **b**.
c1M1: Using r = M x their matrix in **a** and **b**.
c2M1: Getting an expression for l_2 with at least one component correct.
c1A1: cao all three components correct
c2A1ft: ft their vector, must have **r =** or **(r-c)x** d = 0 need both equation and r.

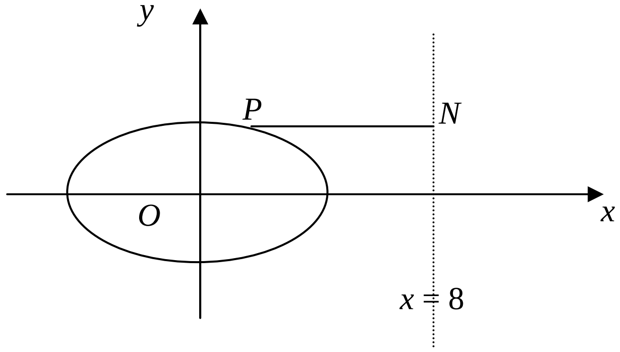
Mark Scheme (Results)

Summer 2013

GCE Further Pure Mathematics 3 (6669/01R)

Question Number	Scheme		Marks
	Foci (±5, 0), Directrices $x = \pm\frac{9}{5}$		
1.	$(\pm)ae = (\pm)5$ and $(\pm)\frac{a}{e} = (\pm)\frac{9}{5}$	Correct equations (ignore ±'s)	B1
	so $e = \frac{5}{a} \Rightarrow \frac{a^2}{5} = \frac{9}{5} \Rightarrow a^2 = 9$ or $a = \frac{5}{e} \Rightarrow \frac{5}{e^2} = \frac{9}{5} \Rightarrow e = \frac{5}{3} \Rightarrow a = 3$	M1: Solves using an appropriate method to find a^2 or a A1: $a^2 = 9$ or $a = (\pm)3$	M1A1
	$b^2 = a^2e^2 - a^2 \Rightarrow b^2 = 25 - 9$ so $b^2 = 16$ $(\Rightarrow b = 4)$ or $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(\frac{25}{9} - 1\right)$ $b^2 = 16$ $(\Rightarrow b = 4)$	M1: Use of $b^2 = a^2(e^2 - 1)$ to obtain a numerical value for b^2 or b A1: : $b^2 = 16$ or $b = (\pm)4$	M1 A1
	So $\frac{x^2}{9} - \frac{y^2}{16} = 1$	M1:Use of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with their a^2 and b^2 A1: Correct hyperbola in any form.	M1 A1
			(7)

Question Number	Scheme		Marks
2.	l_1: $(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ l_2: $(3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + \lambda(-4\mathbf{i} + 6\mathbf{j} + \mathbf{k})$		
(a)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 2 \\ -4 & 6 & 1 \end{vmatrix} = -9\mathbf{i} - 12\mathbf{j} + 36\mathbf{k}$	M1: Correct attempt at a vector product between $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $-4\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ (if the method is unclear then 2 components must be correct) allowing for the sign error in the *y* component. A1: Any multiple of $(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$	M1A1
			(2)
(b) Way 1	$\mathbf{a}_1 - \mathbf{a}_2 = \pm(2\mathbf{i} + 8\mathbf{j} + \mathbf{k})$	M1: Attempt to subtract position vectors A1: Correct vector $\pm(2\mathbf{i} + 8\mathbf{j} + \mathbf{k})$ (Allow as coordinates)	M1 A1
	So $p = \dfrac{\begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -9 \\ -12 \\ 36 \end{pmatrix}}{\sqrt{9^2 + 12^2 + 36^2}}$	Correct formula for the distance using their vectors: $\dfrac{"\pm(2\mathbf{i} + 8\mathbf{j} + \mathbf{k})" \bullet "\mathbf{n}"}{\lvert"\mathbf{n}"\rvert}$	M1
	$p = \dfrac{\pm 78}{\sqrt{1521}} = \dfrac{\pm 78}{39} = 2$	M1: Correctly forms a scalar product in the numerator **and** Pythagoras in the denominator. **(Dependent on the previous method mark)** A1: 2 (**not** -2)	dM1 A1
			(5)
(b) Way 2	$(\mathbf{i} - \mathbf{j} + \mathbf{k}) \bullet (3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) = -13\ (d_1)$ $(3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) \bullet (3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) = 13\ (d_2)$	M1: Attempt scalar product between their **n** and either position vector A1: Both scalar products correct	M1A1
	$\dfrac{\pm 13}{\sqrt{3^2 + 4^2 + 12^2}} (= 1)$	Divides either of their scalar products by the magnitude of their normal vector. $\dfrac{d_1 \text{ or } d_2}{\lvert"\mathbf{n}"\rvert}$	M1
	$p = \dfrac{d_1}{\lvert"\mathbf{n}"\rvert} - \dfrac{d_2}{\lvert"\mathbf{n}"\rvert}$ or $2 \times \dfrac{d_1}{\lvert"\mathbf{n}"\rvert}$	M1: Correct attempt to find the required distance i.e. subtracts their $\dfrac{d_1}{\lvert"\mathbf{n}"\rvert}$ and $\dfrac{d_2}{\lvert"\mathbf{n}"\rvert}$ or doubles their $\dfrac{d_1}{\lvert"\mathbf{n}"\rvert}$ if $\lvert d_1 \rvert = \lvert d_2 \rvert$. **(Dependent on the previous method mark)** A1: 2 (**not** -2)	dM1 A1
			(5)
			Total 7

Question Number	Scheme		Marks
3. (a)		A closed curve approximately symmetrical about both axes. A vertical line to the right of the curve. A horizontal line from any point on the ellipse to the vertical line **with both P and N clearly marked.**	B1 (1)
3. (b)	M is $\left(\frac{x+8}{2}, y\right) = (X, Y)$	M1: Finds the mid-point of PN	M1A1
	or $\left(\frac{6\cos\theta+8}{2}, 3\sin\theta\right) = (X, Y)$	A1: Correct mid-point	
	$\frac{(2X-8)^2}{36} + \frac{Y^2}{9} = 1$	M1: Attempt cartesian equation A1: Correct equation	M1 A1
			(4)
	The next 3 marks are dependent on having the equation of a circle.		
(c)	Circle because equation may be written $(x-4)^2 + y^2 = 3^2$	Convincing argument – allow follow through provided they do have a circle! Can be implied by their centre and radius.	B1ft
	The centre is (4, 0) and the radius is 3	M1: Use their circle equation to find centre and radius A1: Correct centre and radius	M1A1
			(3)
			Total 8
	<u>Special Case:</u> **In (b) they assume the locus is a circle and find the intercepts on the *x*-axis as (1, 0) and (7, 0) and hence deduce the centre (4, 0) and radius 3.** **This approach scores no marks in (b) but allow recovery in (c).**		

Question Number	Scheme		Marks
4.	$\begin{pmatrix}2&0&3\\0&2&-1\\0&1&2\end{pmatrix}\begin{pmatrix}1+s+t\\-1+s+2t\\2\quad-2t\end{pmatrix}$	M1: Writes Π_1 as a single vector A1: Correct statement	M1A1
	$\begin{pmatrix}2&0&3\\0&2&-1\\0&1&2\end{pmatrix}\begin{pmatrix}1+s+t\\-1+s+2t\\2\quad-2t\end{pmatrix}=\begin{pmatrix}2+2s+2t+6-6t\\-2+2s+4t-2+2t\\-1+s+2t+4-4t\end{pmatrix}$		M1A1
	M1: Correct attempt to multiply A1: Correct vector in any form		
	$=\begin{pmatrix}8+2s-4t\\-4+2s+6t\\3+s-2t\end{pmatrix}$	Correct simplified vector	B1
	$\mathbf{r}=\begin{pmatrix}8\\-4\\3\end{pmatrix}+s\begin{pmatrix}2\\2\\1\end{pmatrix}+t\begin{pmatrix}-4\\6\\-2\end{pmatrix}$		
	$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&1\\-4&6&-2\end{vmatrix}=-10\mathbf{i}+20\mathbf{k}$	M1: Attempts cross product of their direction vectors A1: Any multiple of $-10\mathbf{i}+20\mathbf{k}$	M1A1
	(8i – 4j + 3k).(i – 2k) = 8 - 6	Attempt scalar product of their normal vector with their position vector	M1
	r. (i - 2k) = 2	Correct equation (accept any correct equivalent e.g. **r. (-**10**i** + 20**k) =** -20)	A1
			(9)

Question Number	Scheme		Marks
5(a)	$I_n = \left[x^n(2x-1)^{\frac{1}{2}}\right]_1^5 - \int_1^5 nx^{n-1}(2x-1)^{\frac{1}{2}}\,dx$	M1: Parts in the correct direction including a valid attempt to integrate $(2x-1)^{-\frac{1}{2}}$ A1: Fully correct application – may be un-simplified. (Ignore limits)	M1 A1
	$I_n = \underline{5^n \times 3 - 1} - \int_1^5 nx^{n-1}\underline{(2x-1)(2x-1)^{-\frac{1}{2}}}\,dx$	Obtains a correct (possibly un-simplified) expression using the limits 5 and 1 **and** writes $(2x-1)^{\frac{1}{2}}$ as $(2x-1)(2x-1)^{-\frac{1}{2}}$	B1
	$I_n = 5^n \times 3 - 1 - 2nI_n + nI_{n-1}$	Replaces $\int x^n(2x-1)^{-\frac{1}{2}}dx$ with I_n and $\int x^{n-1}(2x-1)^{-\frac{1}{2}}dx$ with I_{n-1}	dM1
	$(2n+1)I_n = nI_{n-1} + 3\times 5^n - 1$ *	Correct completion to printed answer with no errors seen	A1cso
			(5)
(b)	$I_0 = \int_1^5 (2x-1)^{-\frac{1}{2}}dx = \left[(2x-1)^{\frac{1}{2}}\right]_1^5 = 2$	$I_0 = 2$	B1
	$5I_2 = 2I_1 + 74$ and $3I_1 = I_0 + 14$	M1: Correctly applies the given reduction formula twice A1: Correct <u>equations</u> for I_2 and I_1 (may be implied)	M1 A1
	So $I_1 = \frac{16}{3}$ and $I_2 = ...$ or $5I_2 = 2\frac{I_0+14}{3} + 74$ and $I_2 = ...$	Completes to obtains a numerical expression for I_2	dM1
	$I_2 = \frac{254}{15}$		B1
			(5)
			Total 10

Question Number	Scheme		Marks
6. (a)	$\begin{pmatrix} 4 & 2 & 3 \\ 2 & b & 0 \\ a & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ ... \\ ... \end{pmatrix}, \; = \lambda\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \; \lambda = 8$	M1: Multiplies the given matrix by the given eigenvector M1: Equates the x value to λ A1: $\lambda = 8$	M1, M1, A1
			(3)
(b)	$\begin{pmatrix} 8 \\ 2+2b \\ a+2 \end{pmatrix} = "8"\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ So a = -2 and b = 7	M1: Their 2 + 2b = 2λ or their a + 2 = 0 A1: b = 7 **or** a = -2 A1: b = 7 **and** a = -2	M1 A1 A1
			(3)
(c)	$\begin{vmatrix} 4-\lambda & 2 & 3 \\ 2 & 7-\lambda & 0 \\ -2 & 1 & 8-\lambda \end{vmatrix} = 0$ $\therefore (4-\lambda)(7-\lambda)(8-\lambda) - 2\times 2(8-\lambda) + 3(2+2(7-\lambda)) = 0$		M1
	Correct attempt to establish the Characteristic Equation. **= 0 is required but may be implied by later work** **Allow this mark if the equation is in terms of a, b, c**		
	Attempts to factorise i.e. $(8-\lambda)(30-11\lambda+\lambda^2)$ or $(6-\lambda)(40-13\lambda+\lambda^2)$ or $(5-\lambda)(48-14\lambda+\lambda^2)$ (NB $240-118\lambda+19\lambda^2-\lambda^3=0$)		M1 A1
	M1: Attempt to factorise their cubic – an attempt to identify a linear factor and processes to obtain a simplified quadratic factor A1: Correct factorisation into one linear and one quadratic factor		
	Eigenvalues are 5 and 6	M1: Solves their equation to obtain the other eigenvalues A1: 5 and 6	M1 A1
			(5)
			Total 8

Question Number	Scheme		Marks
7(a)	Put 6cosh x = 9 - 2sinh x		M1
	$6\times\frac{1}{2}(e^{x}+e^{-x})=9-2\times\frac{1}{2}(e^{x}-e^{-x})$	Replaces cosh$x$ and sinh$x$ by the **correct** exponential forms	M1
	$4e^{x}+2e^{-x}-9=0 \Rightarrow 4e^{2x}-9e^{x}+2=0$	M1: Multiplies by e^x A1: Correct quadratic in e^x in any form with terms collected	M1 A1
	So $e^{x}=\frac{1}{4}$ or 2 and x = ln2 or $\ln\frac{1}{4}$	M1: Solves their quadratic in e^x A1: Correct values of x (Any correct equivalent form)	M1 A1
			(6)
(b)	Area is $\int(9-2\sinh x-6\cosh x)\,dx$	$\int(9-2\sinh x-6\cosh x)\,dx$ or $\int(6\cosh x-(9-2\sinh x))\,dx$ or the equivalent in exponential form	M1
	$\pm(9x-2\cosh x-6\sinh x)$ or $\pm(9x-4e^{x}+2e^{-x})$	M1: Attempt to integrate A1: Correct integration	M1 A1
	$\pm([9\ln 2-2\cosh\ln 2-6\sinh\ln 2]-[9\ln\frac{1}{4}-2\cosh\ln\frac{1}{4}-6\sinh\ln\frac{1}{4}])$		dM1
	Complete substitution of their limits from part (a). Depends on both previous M's		
	$=\pm(9\ln(2\div\frac{1}{4})-(e^{\ln 2}+e^{-\ln 2})-3(e^{\ln 2}-e^{-\ln 2})+(e^{\ln\frac{1}{4}}+e^{-\ln\frac{1}{4}})+3(e^{\ln\frac{1}{4}}-e^{-\ln\frac{1}{4}}))$		M1
	Combines logs correctly and uses cosh and sinh of ln correctly at least once		
	$\left(9\ln 8-\frac{5}{2}-\frac{18}{4}+4.25-11.25\right)$ =9ln8 –14 or 27ln2 – 14 Any correct equivalent		A1cao
	Subtracting the wrong way round could score 5/6 max		
			(6)
			Total 12
	<u>Note</u> **If they use** $4e^{2x}-9e^{x}+2$ **in (b) to find the area – no marks**		

Question Number	Scheme		Marks
8(a)	$\frac{dy}{dx} = x^{-\frac{1}{2}}$	Correct derivative (may be un-simplified)	B1
	$s = \int \sqrt{1+(x^{-\frac{1}{2}})^2}\,dx = \int_1^8 \sqrt{(1+\frac{1}{x})}\,dx$	A correct formula quoted or implied. There must be some working before the printed answer.	B1
			(2)
(b)	$x = \sinh^2 u \Rightarrow \frac{dx}{du} = 2\sinh u \cosh u$	Correct derivative	B1
	$(1+\frac{1}{x}) = 1 + \text{cosech}^2 u = \coth^2 u$	$(1+\frac{1}{x}) = \coth^2 u$ or $(1+\frac{1}{x}) = \frac{\cosh^2 u}{\sinh^2 u}$ (may be implied by later work)	B1
	$s = \int \coth u . 2\sinh u \cosh u \, du =$ $\int 2\cosh^2 u du$	M1: Complete substitution A1: $\int 2\cosh^2 u du$	M1 A1
	$= u + \frac{1}{2}\sinh 2u$ or $\frac{1}{4}e^{2u} + u - \frac{1}{4}e^{-2u}$	M1: Uses $\cosh 2u = \pm 2\cosh^2 u \pm 1$ or changes to exponentials in an attempt to integrate an expression of the form $k\cosh^2 u$ A1: Correct integration	dM1 A1
	$x = 8 \Rightarrow u = \text{arsinh}\sqrt{8} = \ln(3+2\sqrt{2})$, $x = 1 \Rightarrow u = \text{arsinh}1 = \ln(1+\sqrt{2})$		
	$\left[u + \frac{1}{2}\sinh 2u\right]_{\text{arsinh}1}^{\text{arsinh}\sqrt{8}}$ $= \text{arsinh}\sqrt{8} + \frac{1}{2}\sinh(2\text{arsinh}\sqrt{8}) - (\text{arsinh}1 + \frac{1}{2}\sinh(2\text{arsinh}1))$ *or* $\left[\frac{1}{4}e^{2u} + u - \frac{1}{4}e^{-2u}\right]_{\text{arsinh}1}^{\text{arsinh}\sqrt{8}}$ $= \frac{1}{4}e^{\text{arsinh}\sqrt{8}} + \text{arsinh}\sqrt{8} - \frac{1}{4}e^{-2\text{arsinh}1}$ *or* $\left[\text{arsinh}\sqrt{x} + \frac{1}{2}\sinh(2\text{arsinh}\sqrt{x})\right]_1^8$ $= \text{arsinh}\sqrt{8} + \frac{1}{2}\sinh(2\text{arsinh}\sqrt{8}) - (\text{arsinh}1 + \frac{1}{2}\sinh(2\text{arsinh}1))$		ddM1A1
	M1: The limits $\text{arsinh}\sqrt{8}$ and $\text{arsinh}1$ or their $\ln(3+2\sqrt{2})$ and $\ln(1+\sqrt{2})$ used correctly in their f(u) or the limits 8 and 1 used correctly if they revert to x **Dependent on both previous M's** A1: A completely correct expression		
	$\ln(1+\sqrt{2}) + 5\sqrt{2}$		A1
			(9)
			Total 11

Mark Scheme (Results)

Summer 2013

GCE Further Pure Mathematics 3 (6669/01)

Question Number	Scheme		Marks
	Mark (a) and (b) together		
1. (a) & (b)	$ae = 13$ **and** $a^2(e^2 - 1) = 25$	Sight of **both** of these (can be implied by their work) (allow $\pm$ ae $= \pm 13$ or $\pm$ ae $= 13$ or ae $= \pm 13$)	B1
	Solves to obtain $a^2 = ...$ or $a = ...$	Eliminates *e* to reach $a^2 = ...$or $a = ...$	M1
	$a = 12$	Cao (not ± 12) unless -12 is rejected	A1
	e = 13/ "12"	Uses their *a* to find *e* or finds *e* by eliminating *a* (Ignore $\pm$ here) (Can be implied by a correct answer)	M1
	$x = (\pm)\dfrac{a}{e},\ = \pm\dfrac{144}{13}$	M1: $(x =)(\pm)\dfrac{a}{e}$ $\pm$ not needed for this mark nor is *x* and even allow y = $(\pm)\dfrac{a}{e}$ here – just look for use of $\dfrac{a}{e}$ with numerical *a* and *e*.	M1, A1
		A1: $x = \pm\dfrac{144}{13}$ oe but must be an <u>equation</u> (Do not allow $x = \pm\dfrac{12}{13/12}$)	
			Total 6
	If they use the eccentricity equation for the ellipse ($b^2=a^2(1-e^2)$) allow the M's		

Question Number	Scheme	Marks
2. (a)	$k\,\text{arsinh}\left(\frac{2x}{3}\right)\ (+c)$ or $k\ln[px+\sqrt{(p^2x^2+\frac{9}{4}p^2)}]\ (+c)$	M1
	$\frac{1}{2}\text{ar}\sinh\left(\frac{2x}{3}\right)\ (+c)$ or $\frac{1}{2}\ln[px+\sqrt{(p^2x^2+\frac{9}{4}p^2)}]\ (+c)$	A1
		(2)
(b)	So: $\frac{1}{2}\ln\left[6+\sqrt{45}\right]-\frac{1}{2}\ln\left[-6+\sqrt{45}\right]=\frac{1}{2}\ln\left[\frac{6+\sqrt{45}}{-6+\sqrt{45}}\right]$	M1
	Uses correct limits and combines logs	
	$=\frac{1}{2}\ln\left[\frac{6+\sqrt{45}}{-6+\sqrt{45}}\right]\left[\frac{6+\sqrt{45}}{6+\sqrt{45}}\right]=\frac{1}{2}\ln\left[\frac{(6+\sqrt{45})^2}{9}\right]$	M1
	Correct method to rationalise denominator (may be implied) **Method must be clear if answer does not follow their fraction**	
	$=\ln[2+\sqrt{5}]$ (or $\frac{1}{2}\ln[9+4\sqrt{5}]$)	A1cso
	Note that the last 3 marks can be scored without the need to rationalise e.g. $2\times\frac{1}{2}\left[\ln[2x+\sqrt{(4x^2+9)}]\right]_0^3=\ln(6+\sqrt{45})-\ln 3=\ln(\frac{6+\sqrt{45}}{3})$ M1: Uses the limits 0 and 3 and doubles M1: Combines Logs A1: $\ln[2+\sqrt{5}]$ oe	
		(3)
		Total 5
Alternative for **(a)**	$x=\frac{3}{2}\sinh u\Rightarrow\int\frac{1}{\sqrt{9\sinh^2 u+9}}.\frac{3}{2}\cosh u\,\text{d}u=k\,\text{arsinh}\left(\frac{2x}{3}\right)\ (+c)$	M1
	$\frac{1}{2}\text{ar}\sinh\left(\frac{2x}{3}\right)\ (+c)$	A1
Alternative for **(b)**	$\left[\frac{1}{2}\text{arsinh}\left(\frac{2x}{3}\right)\right]_{-3}^{3}=\frac{1}{2}\text{arsinh}\ \ 2\ -\frac{1}{2}\text{arsinh}\ \ -2$	
	$\frac{1}{2}\ln(2+\sqrt{5})-\frac{1}{2}\ln(\sqrt{5}-2)=\frac{1}{2}\ln(\frac{2+\sqrt{5}}{\sqrt{5}-2})$	M1
	Uses correct limits and combines logs	
	$=\frac{1}{2}\ln(\frac{2+\sqrt{5}}{\sqrt{5}-2}.\frac{\sqrt{5}+2}{\sqrt{5}+2})=\frac{1}{2}\ln(\frac{2\sqrt{5}+4+5+2\sqrt{5}}{5-4})$	M1
	Correct method to rationalise denominator (may be implied) **Method must be clear if answer does not follow their fraction**	
	$=\frac{1}{2}\ln[9+4\sqrt{5}]$	A1cso

Question Number	Scheme		Marks
3.	$(\frac{dx}{d\theta}) = 2\sinh 2\theta$ **and** $(\frac{dy}{d\theta}) = 4\cosh\theta$ Or equivalent correct derivatives		B1
	$A = (2\pi)\int 4\sinh\theta\sqrt{"2\sinh 2\theta"^2 + "4\cosh\theta"^2}\,d\theta$ or $A = (2\pi)\int 4\sinh\theta\sqrt{(1+(\frac{"4\cosh\theta"}{"2\sinh 2\theta"})^2}\,.2\sinh 2\theta\,d\theta$		M1
	Use of correct formula including replacing dx with $"2\sinh 2\theta"d\theta$ if chain rule used. Allow the omission of the 2π here.		
	$A = 32\pi\int \sinh\theta\cosh^2\theta\,d\theta$ $A = 32\pi\int(\sinh\theta + \sinh^3\theta)\,d\theta$		B1
	<u>Completely correct</u> expression for *A* with the square root removed **This mark may be recovered later if the 2π is introduced later**		
	$A = \frac{32\pi}{3}\left[\cosh^3\theta\right]_0^1$	M1: Valid attempt to integrate a correct expression or a multiple of a correct expression – dependent on the first M1 A1: Correct expression	dM1A1
	$= \frac{32\pi}{3}\left[\cosh^3 1 - 1\right]$	M1: Uses the limits 0 and 1 correctly. Dependent on **both** previous M"s A1: Cao and cso (no errors seen)	ddM1A1
			(7)

	Example Alternative Integration for last 4 marks		
	$\int\sinh\theta\cosh^2\theta\,d\theta = \int\sinh\theta(1+\sinh^2\theta)\,d\theta = \int(\sinh\theta+\sinh^3\theta)\,d\theta$ $\int(\sinh\theta + \frac{1}{4}\sinh 3\theta - \frac{3}{4}\sinh\theta)\,d\theta = \frac{1}{4}\int(\sinh\theta + \sinh 3\theta)\,d\theta$ $= \frac{1}{4}\cosh\theta + \frac{1}{12}\cosh 3\theta$ **dM1:** $\int\sinh\theta\cosh^2\theta\,d\theta = p\cosh\theta + q\cosh 3\theta$ **A1**: $32\pi\left[\frac{1}{4}\cosh\theta + \frac{1}{12}\cosh 3\theta\right]$		dM1A1
	$A = 8\pi\left[\cosh\theta + \frac{1}{3}\cosh 3\theta\right]_0^1$ $= 8\pi(\cosh 1 + \frac{1}{3}\cosh 3 - \cosh 0 - \frac{1}{3}\cosh 0)$ .... $\frac{32\pi}{3}\left[\cosh^3 1 - 1\right]$	M1: Uses the limits 0 and 1 correctly. Dependent on **both** previous M"s A1: Cao	ddM1A1

Question Number	Scheme		Marks
3.	**Alternative Cartesian Approach**		
	$x = 1 + \frac{y^2}{8}$	Any correct Cartesian equation	B1
	$\frac{dx}{dy} = \frac{y}{4}$ or $\frac{dy}{dx} = \frac{\sqrt{2}}{(x-1)^{\frac{1}{2}}}$	Correct Derivative	B1
	$A = \int 2\pi . y \sqrt{(1 + \left(\frac{y}{4}\right)^2})\mathrm{d}y$ or $A = \int 2\pi . \sqrt{8}(x-1)^{\frac{1}{2}} \sqrt{(1 + \left(\frac{2}{x-1}\right)})\mathrm{d}x$		M1
	Use of a correct formula		
	$A = 2\pi \times \frac{2}{3} \times 8 \left(1 + \frac{y^2}{16}\right)^{\frac{3}{2}}$ or $A = \frac{4\pi\sqrt{8}}{3}\ \ x+1\ ^{\frac{3}{2}}$		dM1 A1
	M1: Convincing attempt to integrate a relevant expression – dependent on the first M1 but allow the omission of 2π		
	A1: Completely correct expression for A		
	$A = 2\pi \times \frac{2}{3} \times 8\ \ 1 + \sinh^2 1\ ^{\frac{3}{2}} - 2\pi \times \frac{2}{3} \times 8$ or $2\pi \times \frac{2}{3} \times \sqrt{8}\ \ 1 + \cosh 2\ ^{\frac{3}{2}} - \frac{32\pi}{3}$		ddM1
	Correct use of limits (0 → 4sinh1 for y or 1 → cosh2 for *x*)		
	Use $1 + \sinh^2 1 = \cosh^2 1$ to give $\frac{32\pi}{3}\left[\cosh^3 1 - 1\right]$	Use $\cosh 2 = 2\cosh^2 1 - 1$ to give $\frac{32\pi}{3}\left[\cosh^3 1 - 1\right]$	A1

Question Number	Scheme		Marks
4.	$\frac{dy}{dx} = \frac{40}{\sqrt{(x^2-1)}}$ - 9	M1: $\frac{dy}{dx} = \frac{p}{\sqrt{(x^2-1)}} - q$ A1: Cao	M1 A1
	Put $\frac{dy}{dx}$ **= 0** and obtain $x^2 = ...$ **(Allow sign errors only)**	e.g. $\left(\frac{1681}{81}\right)$	dM1
	$x = \frac{41}{9}$	M1: Square root A1: $x = \frac{41}{9}$ or exact equivalent (not $\pm\frac{41}{9}$)	M1 A1
	$y = 40\ln\{(\frac{41}{9}) + \sqrt{(\frac{41}{9})^2 - 1}\} - "41"$	Substitutes $x = "\frac{41}{9}"$ into the curve and uses the logarithmic form of arcosh	M1
	So $y = 80\ln 3 - 41$	Cao	A1
			Total 7

Question Number	Scheme		Marks
5. (a) (i)&(ii)	$\begin{pmatrix}1&1&a\\2&b&c\\-1&0&1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1+a\\b+c\\1\end{pmatrix}=\lambda_1\begin{pmatrix}0\\1\\1\end{pmatrix}$, and so a = -1, $\lambda_1 = 1$		M1, A1, A1
	M1: Multiplies out matrix with first eigenvector and puts equal to λ_1 times eigenvector. A1 : Deduces a = -1. A1: Deduces λ_1 = 1		
	$\begin{pmatrix}1&1&a\\2&b&c\\-1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}1-a\\2-c\\-2\end{pmatrix}=\lambda_2\begin{pmatrix}1\\0\\-1\end{pmatrix}$, and so $c=2$, $\lambda_2 = 2$		M1, A1, A1
	M1: Multiplies out matrix with second eigenvector and puts equal to λ_2 times eigenvector. A1: Deduces c = 2. A1: Deduces λ_2 = 2		
	$b+c=\lambda_1$ so b = -1	M1: Uses $b+c=\lambda_1$ with their λ_1 to find a value for b (They must have an equation in b and c from the first eigenvector to score this mark) A1: b = -1	M1A1
	(a = -1, b = -1, c = 2, λ_1=1, λ_2 = 2)		(8)
(b)(i)	detP = -d - 1	Allow 1 – d – 2 or 1 – (2 + d) A correct (possibly un-simplified) determinant	B1
(ii)	$\mathbf{P}^T=\begin{pmatrix}1&2&-1\\1&1&0\\0&d&1\end{pmatrix}$ **or** minors $\begin{pmatrix}1&d+2&1\\1&1&1\\d&d&-1\end{pmatrix}$ **or** cofactors $\begin{pmatrix}1&-2-d&1\\-1&1&-1\\d&-d&-1\end{pmatrix}$ a **correct** first step		B1
	$\dfrac{1}{-d-1}\begin{pmatrix}1&-1&d\\-2-d&1&-d\\1&-1&-1\end{pmatrix}$	M1: Identifiable full attempt at inverse **including reciprocal of determinant**. Could be indicated by at least 6 correct elements. A1: Two rows or two columns correct (ignoring determinant) **BUT M0A1A0 or M0A1A1 is not possible** A1: Fully correct inverse	M1 A1 A1
			(5)
			Total 13

Question Number	Scheme		Marks
6(a)	$I_n = \int_0^4 \underline{x^{n-1} \times x(16-x^2)^{\frac{1}{2}}}\mathrm{d}x$	M1: Obtains $x(16-x^2)^{\frac{1}{2}}$ prior to integration A1: Correct underlined expression (can be implied by their integration)	M1A1
	$I_n = \left[-\frac{1}{3}x^{n-1}(16-x^2)^{\frac{3}{2}}\right]_0^4 + \frac{n-1}{3}\int_0^4 x^{n-2}(16-x^2)^{\frac{3}{2}}\mathrm{d}x$		dM1
	dM1: Parts in the correct direction (Ignore limits)		
	$\therefore I_n = \frac{n-1}{3}\int_0^4 x^{n-2}(16-x^2)(16-x^2)^{\frac{1}{2}}\mathrm{d}x$		
	i.e. $I_n = \frac{16(n-1)}{3}I_{n-2} - \frac{n-1}{3}I_n$	Manipulates to obtain at least one integral in terms of I_n or I_{n-2} on the rhs.	M1
	$I_n(1+\frac{n-1}{3}) = \frac{16(n-1)}{3}I_{n-2}$	Collects terms in I_n from both sides	M1
	$(n+2)I_n = 16(n-1)I_{n-2}$ *	Printed answer with no errors	A1*cso
			(6)
Way 2	$\int_0^4 x^n(16-x^2)^{\frac{1}{2}}\mathrm{d}x = \int_0^4 x^n\frac{(16-x^2)}{(16-x^2)^{\frac{1}{2}}}\mathrm{d}x = \int_0^4 \frac{16x^n}{(16-x^2)^{\frac{1}{2}}}\mathrm{d}x - \int_0^4 \frac{x^{n+2}}{(16-x^2)^{\frac{1}{2}}}\mathrm{d}x$		
	$= \int_0^4 16x^{n-1}\times x(16-x^2)^{-\frac{1}{2}}\mathrm{d}x - \int_0^4 x^{n+1}\times x(16-x^2)^{-\frac{1}{2}}\mathrm{d}x$		M1A1
	M1: Obtains $x(16-x^2)^{-\frac{1}{2}}$ prior to integration A1: Correct expressions		
	$= \left[-16x^{n-1}(16-x^2)^{\frac{1}{2}}\right]_0^4 + 16(n-1)\int_0^4 x^{n-2}(16-x^2)^{\frac{1}{2}}\mathrm{d}x$ $-(\left[-x^{n+1}(16-x^2)^{\frac{1}{2}}\right]_0^4 + (n+1)\int_0^4 x^n(16-x^2)^{\frac{1}{2}}\mathrm{d}x)$		dM1
	dM1: Parts in the correct direction on both (Ignore limits)		
	$I_n = 16(n-1)I_{n-2} - (n+1)I_n$	Manipulates to obtain at least one integral in terms of I_n or I_{n-2} on the rhs.	M1
	$I_n(1+n+1) = 16(n-1)I_{n-2}$	Collects terms in I_n from both sides	M1
	$(n+2)I_n = 16(n-1)I_{n-2}$ *	Printed answer with no errors	A1*
Way 3	$\int_0^4 x^n(16-x^2)^{\frac{1}{2}}\mathrm{d}x = \int_0^4 x\times x^{n-1}\frac{(16-x^2)}{(16-x^2)^{\frac{1}{2}}}\mathrm{d}x$	M1: Obtains $x(16-x^2)^{-\frac{1}{2}}$ prior to integration A1: Correct expression	M1A1
	$= \left[-x^{n-1}(16-x^2)(16-x^2)^{\frac{1}{2}}\right]_0^4 + \int_0^4(16(n-1)x^{n-2} - (n+1)x^n)(16-x^2)^{\frac{1}{2}}\mathrm{d}x$		dM1
	dM1: Parts in the correct direction (Ignore limits)		
	$I_n = 16(n-1)I_{n-2} - (n+1)I_n$	Manipulates to obtain at least one integral in terms of I_n or I_{n-2} on the rhs.	M1
	$I_n(1+n+1) = 16(n-1)I_{n-2}$	Collects terms in I_n from both sides	M1
	$(n+2)I_n = 16(n-1)I_{n-2}$ *	Printed answer with no errors	A1*

Question Number	Scheme		Marks
(b)	$I_1 = \int_0^4 x\sqrt{(16-x^2)}\,dx = \left[-\tfrac{1}{3}(16-x^2)^{\frac{3}{2}}\right]_0^4 = \dfrac{64}{3}$	M1: Correct integration to find I_1 A1: $\dfrac{64}{3}$ or equivalent (May be implied by a later work – they are not asked explicitly for I_1)	M1 A1
	$\dfrac{64}{3}$ must come from correct work		
	Using $x = 4\sin\theta$: $I_1 = \int_0^{\frac{\pi}{2}} 4\sin\theta\sqrt{(16-16\sin^2\theta)}\,4\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} 64\sin\theta\cos^2\theta\,d\theta$ $= \left[-\dfrac{64}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$ M1: A <u>complete</u> substitution and attempt to substitute <u>changed</u> limits A1: $\dfrac{64}{3}$ or equivalent		
	$I_5 = \dfrac{64}{7}I_3,\ I_3 = \dfrac{32}{5}I_1$	Applies to apply reduction formula twice. First M1 for I_5 in terms of I_3, second M1 for I_3 in terms of I_1 (Can be implied)	M1, M1
	$I_5 = \dfrac{131072}{105}$	Any **<u>exact</u>** equivalent (Depends on all previous marks having been scored)	A1
			(5)
			Total 11

Question Number	Scheme		Marks
7(a)	$\left(\dfrac{dx}{d\theta}=-a\sin\theta\right.$ and $\dfrac{dy}{d\theta}=b\cos\theta\left.\right)$ so $\dfrac{dy}{dx}=\dfrac{b\cos\theta}{-a\sin\theta}$		M1 A1
	M1: Differentiates both *x* and *y* and divides correctly A1: Fully correct derivative		
	Alternative: M1: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\Rightarrow\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0\Rightarrow y'=-\dfrac{b^2x}{a^2y}=-\dfrac{b^2a\cos\theta}{a^2b\sin\theta}$ Differentiates implicitly **and** substitutes for *x* and *y* A1: $=-\dfrac{b\cos\theta}{a\sin\theta}$		
	Normal has gradient $\dfrac{a\sin\theta}{b\cos\theta}$ *or* $\dfrac{a^2y}{b^2x}$	Correct perpendicular gradient rule	M1
	$(y-b\sin\theta)=\dfrac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$	Correct straight line method using a „changed" gradient which is a function of θ	M1
	If $y = mx + c$ is used need to find *c* for M1		
	$ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta$ *		A1
	Fully correct completion to printed answer		
			(5)
(b)	$x=\dfrac{(a^2-b^2)\cos\theta}{a}$	Allow un-simplified	B1
	$y=-\dfrac{(a^2-b^2)\sin\theta}{b}$	Allow un-simplified	B1
	$\left(=\tfrac{1}{2}\dfrac{(a^2-b^2)^2\cos\theta\sin\theta}{ab}\right)$	$=\tfrac{1}{4}\dfrac{(a^2-b^2)^2}{ab}\sin 2\theta$	M1A1
	M1: Area of triangle is $\tfrac{1}{2}$"OA"$\times$"OB" and uses double angle formula correctly **A1: Correct expression for the area (must be positive)**		
			(4)
(c)	Maximum area when $\sin 2\theta=1$ so $\theta=\dfrac{\pi}{4}$ or 45	Correct value for θ (may be implied by correct coordinates)	B1
	So the point *P* is at $\left(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}}\right)$ oe $\left(a\cos\dfrac{\pi}{4},\ b\sin\dfrac{\pi}{4}\right)$ scores B1M1A0	M1: Substitutes their value of θ where $0<\theta<\dfrac{\pi}{2}$ or $0<\theta<90$ into their parametric coordinates A1: Correct exact coordinates	M1 A1
	Mark part (c) independently		
			(3)
			Total 12

Question Number	Scheme		Marks
8(a)	$(\mathbf{6i}+\mathbf{2j}+\mathbf{12k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=34$	Attempt scalar product	M1
	$\left\lvert\dfrac{(\mathbf{6i}+\mathbf{2j}+\mathbf{12k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})-5}{\sqrt{3^2+4^2+2^2}}\right\rvert$	Use of correct formula	M1
	$\sqrt{29}$ (not $-\sqrt{29}$)	Correct distance (Allow $29/\sqrt{29}$)	A1
			(3)
(a) Way 2	$\mathbf{r}=(\mathbf{6i}+\mathbf{2j}+\mathbf{12k})+\lambda(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})$ $\therefore\ 6+3\lambda\ \ 3+\ \ 2-4\lambda\ \ \ -4\ +\ 12+2\lambda\ \ 2=5$		M1
	Substitutes the parametric coordinates of the line through (6, 2, 12) perpendicular to the plane into the cartesian equation.		
	$\lambda=-1\Rightarrow\ 3,6,10$ or $-3\mathbf{i}+4\mathbf{j}-2\mathbf{k}$	Solves for λ to obtain the required point or vector.	M1
	$\sqrt{29}$	Correct distance	A1
(a) Way 3	**Parallel plane** containing (6, 2, 12) is $\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=34$ $\Rightarrow\dfrac{\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})}{\sqrt{29}}=\dfrac{34}{\sqrt{29}}$	Origin to this plane is $\dfrac{34}{\sqrt{29}}$	M1
	$\Rightarrow\dfrac{\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})}{\sqrt{29}}=\dfrac{5}{\sqrt{29}}$	Origin to plane is $\dfrac{5}{\sqrt{29}}$	M1
	$\dfrac{34}{\sqrt{29}}-\dfrac{5}{\sqrt{29}}=\sqrt{29}$	Correct distance	A1
(b) **For a cross product, if the method is unclear, 2 out of 3 components should be correct for M1**	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 5\\ 1 & -1 & -2\end{vmatrix}=\begin{pmatrix}3\\9\\-3\end{pmatrix}$	M1: Attempts $(\mathbf{2i}+\mathbf{1j}+\mathbf{5k})\times(\mathbf{i}-\mathbf{j}-2\mathbf{k})$ A1: Any multiple of $\mathbf{i}+\mathbf{3j}-\mathbf{k}$	M1A1
	$(\cos\theta)=\dfrac{(\mathbf{3i}-\mathbf{4j}+\mathbf{2k}).(\mathbf{i}+\mathbf{3j}-\mathbf{k})}{\sqrt{3^2+4^2+2^2}\sqrt{1^2+3^2+1^2}}\quad\left(=\dfrac{-11}{\sqrt{29}\sqrt{11}}\right)$		M1
	Attempts scalar product of normal vectors including magnitudes		
	52	Obtains angle using arccos (dependent on previous M1)	dM1 A1
	Do not isw and mark the final answer e.g. 90 – 52 = 38 loses the A1		(5)
(c)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & -1\\ 3 & -4 & 2\end{vmatrix}=\begin{pmatrix}2\\-5\\-13\end{pmatrix}$	M1: Attempt cross product of normal vectors A1: Correct vector	M1A1
	$x=0:(0,\dfrac{5}{2},\dfrac{15}{2}),\quad y=0:(1,\ 0,\ 1),\quad z=0:(\dfrac{15}{13},\dfrac{-5}{13},\ 0)$		M1A1
	M1: Valid attempt at a point on both planes. A1: Correct coordinates **May use way 3 to find a point on the line**		
	$\mathbf{r}\times(\mathbf{-2i}+\mathbf{5j}+\mathbf{13k})=\mathbf{-5i}-\mathbf{15j}+\mathbf{5k}$	M1: $\mathbf{r}\times$ dir = pos.vector $\times$ dir **(This way round)** A1: Correct equation	M1A1
			(6)

Question Number	Scheme	Marks
(c) Way 2	"$x + 3y - z = 0$" and $3x - 4y + 2z = 5$ uses their cartesian form of and eliminate x, or y or z and substitutes back to obtain two of the variables in terms of the third	M1
	$(x = 1 - \frac{2}{5}y$ and $z = 1 + \frac{13}{5}y)$ or $(y = \frac{5z-5}{13}$ and $x = \frac{15-2z}{13})$ or $(y = \frac{5-5x}{2}$ and $z = \frac{15-13x}{2})$	A1
	Cartesian Equations: $x = \dfrac{y-\frac{5}{2}}{-\frac{5}{2}} = \dfrac{z-\frac{15}{2}}{-\frac{13}{2}}$ or $\dfrac{x-1}{-\frac{2}{5}} = y = \dfrac{z-1}{\frac{13}{5}}$ or $\dfrac{x-\frac{15}{13}}{-\frac{2}{13}} = \dfrac{y+\frac{5}{13}}{\frac{5}{13}} = z$	
	Points and Directions: Direction can be any multiple $(0, \frac{5}{2}, \frac{15}{2}), \mathbf{i} - \frac{5}{2}\mathbf{j} - \frac{13}{2}\mathbf{k}$ or $(1,0,1), -\frac{2}{5}\mathbf{i} + \mathbf{j} + \frac{13}{5}\mathbf{k}$ or $(\frac{15}{13}, -\frac{5}{13}, 0), -\frac{2}{13}\mathbf{i} + \frac{5}{13}\mathbf{j} + \mathbf{k}$	M1 A1
	M1:Uses their Cartesian equations correctly to obtain a point and direction A1: Correct point and direction – it may not be clear which is which – i.e. look for the correct numbers either as points or vectors	
	Equation of line in required form: e.g. $\mathbf{r} \times (\mathbf{-2i + 5j + 13k}) = \mathbf{-5i - 15j + 5k}$ Or Equivalent	M1 A1
		(6)
		Total 14

Question Number	Scheme		Marks
(c) Way 3	$\begin{pmatrix} 2\lambda + \mu \\ \lambda - \mu \\ 5\lambda - 2\mu \end{pmatrix} . \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 5 \Rightarrow 12\lambda + 3\mu = 5$	M1: Substitutes parametric form of $\prod_2$ into the vector equation of $\prod_1$ A1: Correct equation	M1A1
	$\mu = \frac{5}{3}, \lambda = 0$ gives $(\frac{5}{3}, -\frac{5}{3}, \frac{10}{3})$ $\mu = 0, \lambda = \frac{5}{12}$ gives $(\frac{5}{6}, \frac{5}{12}, \frac{25}{12})$ Direction $\begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix}$	M1: Finds 2 points and direction A1: Correct coordinates and direction	M1A1
	Equation of line in required form: e.g. $\mathbf{r} \times (\mathbf{-2i + 5j + 13k}) = \mathbf{-5i - 15j + 5k}$ Or Equivalent		M1A1
	Do not allow 'mixed' methods – mark the best single attempt		
	NB for checking, a general point on the line will be of the form: (1 - 2λ, 5λ, 1 + 13λ)		